# SELECTED PROBLEMS AND SOLUTIONS (2017-2021)

A Compllimentary Booklet For All Participants

## KANGAROO MATH COMPETITION

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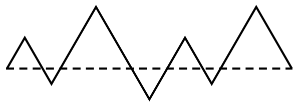
Published in 2022

# Cadet

In the diagram, the dashed line and the black path form seven equilateral triangles. The length of the dashed line is 20. What is the length of the black path?



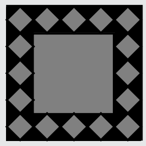
(A) 25 (B) 30 (C) 35 (D) 40 (E) 45

**Answer.** D

Every time the black path and the dashed line intersect, they form an equilateral triangle. Notice that the black path represents two sides of the triangle, while the dashed line represents the other side. This means that the length of the black path is twice the length of the dashed line for every triangle. This occurs throughout the black path, hence its total length is twice the length of the dashed line, which is, 20 × 2 = 40.

## CADET 2017 Q23

Olesia s tablecloth has a regular pattern, as shown in the diagram. What percentage of the tablecloth is black?

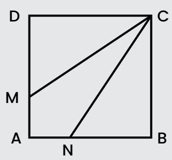


(A) 16 (B) 24 (C) 25 (D) 32 (E) 36

**Answer.** D

Suppose the small square has sides 1, then its diagonal is $\sqrt{1^2+1^2}=\sqrt{2}$ by Pythagorean theorem. It follows that the sides of medium and large square to be $3\sqrt{2}$ and $5\sqrt{2}$, respectively. Then there are $1-\frac{16.1^2+(3\sqrt{2})^2}{(5\sqrt{2})^2}\times 100\% = 32\%$ of the region is black.

Square *ABCD* has sides of length 3 cm. Lines *CM* and *CN* split the square into three piece with the same area. What is the length of *DM*?




(A) 0.5 cm (B) 1 cm (C) 1.5 cm (D) 2 cm (E) 2.5 cm

**Answer.** D

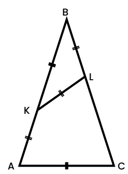
Since CN and CM partitioned the square ABCD into three equiarea regions, so.

$$S_{\Delta CDM} = \frac{1}{2}\cdot CD \cdot DM = \frac{1}{3}\cdot 3^2$$

implies that DM = 2 cm.

## CADET 2018 Q24

In isosceles triangle ABC, points K and L are marked on sides AB and BC respectively so that AK = KL = LB and KB = AC. What is the size of angle ABC?




(A) 30° (B) 35° (C) 36° (D) 40° (E) 44°

**Answer.** C

Connect line CK. Since CL = BC - BL = AB -AK = BK, AK = AL and CK as a common side, so $\Delta AKC \cong \Delta LKC$ by SSS property. If x = $\angle ABC$, then 2x = $\angle KLC = \angle KAC = \angle BCA$ as x = $\angle BKL$. This follows that 5x = $\angle A + \angle B + \angle C = 180°$, thus x = 36°.

## CADET 2019 Q9

Andrew divided some apples into six equal piles. Boris divided the same number of apples into five equal piles. Boris noticed that each of his piles contains two more apples than each of Andrew's piles. How many apples does Andrew have?

(A) 60 (B) 65 (C) 70 (D) 75 (E) 80

**Answer.** A

Suppose that they both have x apples, then $\frac{x}{6}+2=\frac{x}{5}$ Solving for x gives x = 60.

# Cadet

## CADET 2019 Q24

A train is made up of 18 carriages. There are 700 passengers traveling on the train. In any block of five adjacent carriages, there are 199 passengers in total. How many passengers are in the middle two carriages of the train?

(A) 70 (B) 77 (C)78 (D) 96 (E) 103

**Answer.** D

Numbered the carriages from 1 to 18. Consider any six adjacent carriages, says 1–6, then the number of passengers in carriages 1–5 and 2–6 are equal, implies that the first and sixth carriage hold the same number of passengers. Similarly, we get to know that the number of passengers in each carriage has a period of 5. Then the number of passengers in the middle two carriages of the train and in the fourth and fifth carriages of the train are equal, given by $199 \times 4 - 700 = 96$.

## CADET 2020 Q8

Kanga wants to multiply three different numbers from the following list: -5, -3, -1, 2, 4, and 6. What is the smallest result she could obtain?

(A) -200 (B) -120 (C) -90 (D) -48 (E) -15

**Answer.** B

To minimize the product of three distinct numbers, we either take 1 or 3 negative numbers with 2 or 0 positive numbers whose have the largest absolute value. In this case, we can take $(-5) \times 4 \times 6 = -120$.

## CADET 2020 Q23

Sophia has 52 identical isosceles right-angled triangles. She wants to make a square using some of them. How many different sized squares can she make?

(A) 6 (B) 7 (C) 8 (D) 9 (E) 10

**Answer.** C

Suppose the isosceles right-angled triangles have legs of length 1 and hypotenuse of length $\sqrt{2}$, then squares of sides 1 and $\sqrt{2}$ are able to made by her using 2 and 4 identical isosceles right-angled triangles. Since $\frac{52}{2} = 26$ and $\frac{52}{4} = 13$, so Sophia can make squares of sides 2, 3, 4 and 5 using 4, 9, 16 and 25 squares of side 1; and squares of sides $2\sqrt{2}$ and $3\sqrt{2}$ using 4 and 9 squares of side $\sqrt{2}$. This concludes that she can only make 8 squares of distinct sides.

## CADET 2021 Q17

There are 20 questions in a quiz. Each correct answer scores 7 points, each wrong answer scores -4 points, and each question left blank scores 0 points. Eric took the quiz and scored 100 points. How many questions did he leave blank?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

**Answer.** B

Suppose that Eric has made x correct questions, y blank questions and z incorrect questions, then $x + y + z = 20$ and $7x - 4z = 100$ are satisfied. Since $x \leq 20$, so $100 = 7x - 4z \leq 7x$ implies that $x \geq \frac{100}{7}$. He may correct 15, 16, $\cdots$, or 20 questions. After the trials and errors, we have found out that $x = 16$ and $z = 3$, as other x give non–integral y, so $y = 1$.

## CADET 2021 Q21

A soccer ball is made of white hexagons and black pentagons, as seen in the picture. There are a total of 12 pentagons. How many hexagons are there?

(A) 12 (B) 15 (C) 18 (D) 20 (E) 24

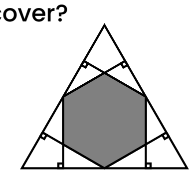
**Answer.** D

Note that every hexagon shares 3 sides with 3 hexagons each, so the number of hexagons in a soccer ball is $\frac{12 \times 5}{3} = 20$. Alternatively, note that each vertex of a pentagon is shared among 1 pentagon and 2 hexagons, so the number of hexagon is given by $\frac{12 \times 5}{3} = 20$.

# Junior

## JUNIOR 2017 Q12

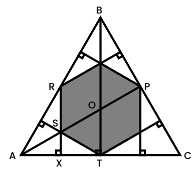
Given an equilateral triangle. From the midpoint of each side, we draw two lines perpen- dicular to the other two sides. The six lines form a hexagon, as in the figure. What fraction of the area of the initial triangle does the hexagon cover?



(A) $\frac{1}{3}$ (B) $\frac{2}{5}$ (C) $\frac{4}{9}$ (D) $\frac{1}{2}$ (E) $\frac{2}{3}$

**Answer.** D

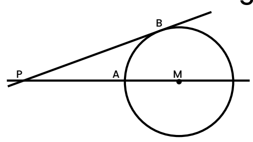
Denote the sides of equilateral triangle be 1 and the center of hexagon be point $O$. Connect lines $AP$ and $BP$ such that $\angle BPA = 90^\circ$.




Since $\Delta ABT \sim \Delta ARX$ by AA similarity, so $\frac{AX}{AT} = \frac{AR}{AB} = \frac{1}{2}$ Thus we have $S\Delta AST = S\Delta OST$ since these two triangles share the same height. Note that $\Delta ABC$ can be partitioned into 6 triangles, which all are congruent to $\Delta AOT$, so $\frac{S_{shaded}}{S_{\Delta ABC}} = \frac{1}{2}$.

## JUNIOR 2017 Q24

Points $A$ and $B$ are on the circle with centre $M$. $PB$ is tangent to the circle at $B$. The distances $PA$ and $MB$ are integers. We know that $PB = PA + 6$. How many possible values are there for the length of $MB$?




(A) 0 (B) 2 (C) 4 (D) 6 (E) 8

**Answer.** D

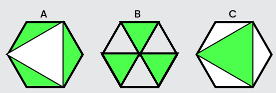
Denote the radius of circle $M$ be $r$ and connect $BM$ such that $AM = BM = r$. Since $\angle PBM = 90^\circ$,

$$PB^2 + BM^2 = PM^2$$
$$(PA + 6)2 + r^2 = (PA + r)^2$$

*by Pythagorean Theorem. Expanding the equation above yields $12 \cdot PA + 36 = 2 \cdot PA \cdot r$. Arrange it in terms of r gives $r = 6 + \frac{18}{PA}$ Since $r \in Z+$, so $PA \mid 18$, showing that $PA = 1, 2, 3, 6, 9, 18$. There are 6 possible values for MB.*

## JUNIOR 2018 Q6

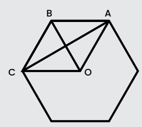
Given three congruent regular hexagons, we call X, Y and Z the total area of the shaded zones in hexagons A, B and C respectively. Which of the following statements is true?




(A) $X = Y = Z$ (B) $Y = Z \neq X$ (C) $X = Z \neq Y$ (D) $X = Y \neq Z$ (E) $X \neq Y \neq Z$

**Answer.** A

We focus on the $\frac{1}{3}$ of the hexagon $ABCDEF$, say $ABCO$.




Since $\Delta ABC \cong \Delta AOC$ by SSS property, so $S_{\Delta ABC} = S_{\Delta AOC}$. Duplicating $\Delta ABC$ and $\Delta AOC$ around point $O$ for $120^\circ$ and $240^\circ$ yield the areas $X$ and $Z$, so $X = Z$. Similarly, $S_{\Delta ABC} = S_{\Delta ABO}$ because these areas occupy half of the parallelogram $ABCO$, so $X = Y$ as the same reason stated.

## JUNIOR 2018 Q24

All numbers in the set $\{1, 2, 3, 4, 5, 6\}$ are written into the cells of a $2 \times 3$ table. How many different ways can this be done such that in each row and in each column the sum of the numbers is divisible by 3?

(A) 36 (B) 42 (C) 45 (D) 48 (E) 56

**Answer.** D

We divide the set into congruence classes modulo 3: $\{1, 4\}$, $\{2, 5\}$ and $\{3, 6\}$. To make the sum of first row to be divisible by 3, we pick one number only from each congruence class modulo 3 and fill in the cells on the row with $6 \times (6 - 2) \times (6 - 2 \times 2) = 48$ ways. Once the first row is done, the second row must be fixed, as the numbers in second row must be $3 - i$ modulo 3 if its upper number is of $i$ modulo 3. This concludes that there are 48 ways to fill in the numbers in the table.

## JUNIOR 2020 Q24

Eight consecutive three-digit positive integers have the following property: each of them is divisible by its last digit. What is the sum of the digits of the smallest of the eight integers?

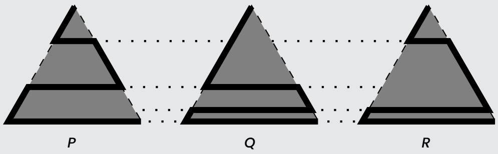
(A) 10 (B) 11 (C) 12 (D) 13 (E) 14

**Answer.** D

Suppose the 3-digit numbers be $\overline{abc}$. If $c \mid \overline{abc}$, then $10 \mid (\overline{abc} - c) = \overline{ab0}$ for all c in either $\{1, 2, \cdots, 8\}$ or $\{2, 3, \cdots, 9\}$. Hence $\overline{ab0}$ must be divisible by 3, 5, 7 and 8, which implies that $\gcd(3, 5, 7, 8) \mid \overline{ab0} \Rightarrow 840 \mid \overline{ab0}$. Therefore, the smallest of the eight integers is 841 and its sum of digits is therefore $8 + 4 + 1 = 13$.

## JUNIOR 2021 Q3

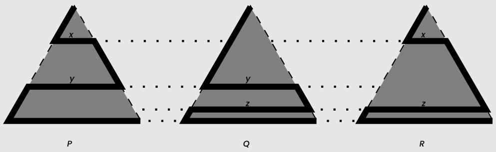
A park is shaped like an equilateral triangle. A cat wants to walk along one of the three indicated paths (thicker lines) from the top corner to the lower right corner. The lengths of the paths are *P*, *Q* and *R*, as shown. Which of the following statements about the lengths of the paths is true?




(A) $P < Q < R$ (B) $P < R < Q$ (C) $P < Q = R$ (D) $P = R < Q$ (E) $P = Q = R$

**Answer.** B

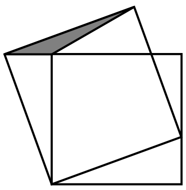
Since the parts of the path on the edges of the equilateral triangles are equal in all cases, so we only need to compare the internal part of the paths. Let the internal parts of the path have lengths *x*, *y* and *z*, respectively.



It is clearly that $x < y < z$, so we have $x + y < x + z < y + z$, then $P < R < Q$.

## JUNIOR 2021 Q24

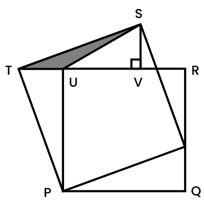
The smaller square in the picture has area 16 and the grey triangle has area 1. What is the area of the larger square?



(A) 17 (B) 18 (C) 19 (D) 20 (E) 21

**Answer.** B

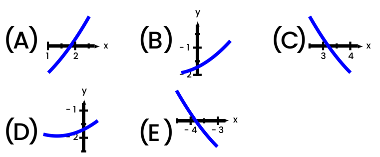
Let *V* be the foot of the perpendicular dropped from *S* to *TR*.




We have $\angle STV = \angle TPU$ since both are equal to $90^\circ - \angle PTU$, so $\Delta STV \cong \Delta TPU$ as both contain a right angle and $PT = ST$. Hence $TU = SV$. The area of the shaded triangle is $\frac{1}{2} \times TU \times SV = \frac{1}{2} \times TU^2 = 1$, so $TU = \sqrt{2}$. The area of the smaller square is 16 so $PU = 4$. Applying Pythagoras' Theorem to $\Delta PTU$ gives $PT^2 = PU^2 + TU^2 = 16 + 2 = 18$, hence the area of the larger square is 18.

## STUDENT 2017 Q5

Four of the following five figures are different parts of the graph of the same quadratic function. Which figure is not part of this graph?



**Answer.** (C)
Since a quadratic function only has increasing and decreasing intervals each, so the choices $(C)$ and $(E)$ reach a contradiction. Since choice $(A)$ shows the increasing pattern of the function over $x \geq 1$, so $(E)$, as the decreasing pattern over $x \leq -3$, must be part of the graph of the function. This shows that $(C)$ is not part of the graph.

## STUDENT 2018 Q8

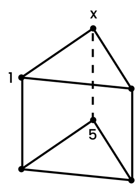
How many ways can the number 1001 be written as the sum of two primes?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

**Solution.** (A)
Since 1001 is an odd integer, so one of the primes must be 2, as other sum of any other two primes give an even integer. This leaves the second prime to be $1001 - 2 = 999$, however, it is not a prime. Hence, there are 0 ways to write 1001 as the sum of two primes.

## STUDENT 2018 Q17

The prism in the figure is formed of two triangles and three squares. The six vertices are numbered from 1 to 6 in such a way that the sum of the four vertices of each square is the same for all three squares. Numbers 1 and 5 are already shown. What number is at the vertex labeled $x$?



(A) 2 (B) 3 (C) 4 (D) 6
(E) The situation is impossible

## STUDENT 2017 Q17

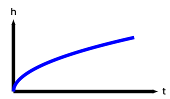
How many positive integers have the property that the number obtained by deleting the last digit is equal to $\frac{1}{14}$ of the original number?

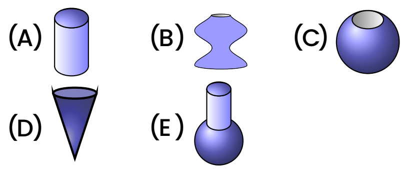
(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

**Answer.** (C)
Let N be the new integer, and a be the deleted last digit, then $n = \frac{1}{14}(10n + a)$ gives $a = 4n$. Since $0 \leq a \leq 9$ and $4 \mid n$, so $(a, n) = (4, 1), (8, 2)$. Thus, there are only two such positive integers, which are 14 and 28.

## STUDENT 2018 Q10

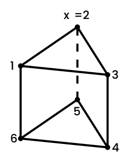
A vase is filled up to the top with water at a constant rate. The graph shows the height h of the water as a function of time t. Which of the following could be the shape of the vase?





**Answer.** (D)
As the rate at which the water level would rise would slow down over time if being filled at a constant rate in the graph above, the cross-sectional area is increasing for greater value of height $h$.

**Answer.** (A)
Adding the numbers in the vertices of 3 squares in the prism gives twice the sum from 1 to 6, so each square gives a sum of $\frac{2}{3} \times (1 + 2 + \cdots + 6) = 14$. Since $14 - 1 - 5 = 8$, and only 2 and 6 can give a sum of 8 in that square containing 1, 5 and $x$, so $x = 2$ or $x = 6$. If $x = 2$, then the other square that contains $x$ and 5 must leave a sum of $14 - x - 5 = 7$ by the other two vertices, which are comprised by the numbers 3 and 4. This forces the number in the lower left vertex must be 6. Below shows an example of a valid numbering.



If $x = 6$, then the other two vertices of the rightmost square must give a sum of $14 - x - 5 = 3$. But 3 can be uniquely expressed as a sum of 1 and 2, and 1 is used as the numbering of the vertex on the leftmost square, hence reach a contradiction. $x = 6$ is therefore rejected.

# Student

## STUDENT 2018 Q18

The roots of the equation $x^2 - x - 2018 = 0$ are $m$ and $n$. What is the value of $n^2 + m$?

(A) 2016 (B) 2017 (C) 2018 (D) 2019 (E) 2020

**Answer.** D

Since n is the root of the quadratic equation $x^2 - x - 2018 = 0$, so $n^2 = n + 2018$. Then $n^2 + m = m + n + 2018 = 1 + 2018 = 2019$ as we have $m + n = 1$ by Viéta formula. Alternatively, solving $x^2 - x - 2018 = 0$ for $x$ yields $m, n = \frac{1 \pm 3\sqrt{897}}{2}$ Then $n^2 + m = \left(\frac{1 \pm 3\sqrt{897}}{2}\right)^2 + \frac{1 \mp 3\sqrt{897}}{2}$ =2019 by brute force calculation.

## STUDENT 2019 Q3

A pyramid has 23 triangular faces. How many edges does this pyramid have?

(A) 23 (B) 24 (C) 46 (D) 48 (E) 69

**Answer.** B

Since the pyramid has 23 triangular faces, so the edges of the faces connects the apex with the vertices of the 23-gon. This shows that it has a total of 23 + 23 = 46 edges.

## STUDENT 2019 Q20

Three different numbers are chosen at random from the set $\{1, 2, 3, \cdots, 10\}$. What is the probability that one of them is the average of the other two?

(A) $\frac{1}{10}$ (B) $\frac{1}{6}$ (C) $\frac{1}{4}$ (D) $\frac{1}{3}$ (E) $\frac{1}{2}$

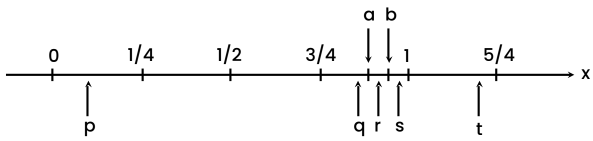
**Answer.** B

To guarantee the mean do exist, we first pick arbitrary two numbers that share the parity with $\binom{1}{2} \times \binom{5}{2}$ ways, then the third number must be fixed as it is determined by the mean of first $\frac{\binom{2}{1} \times \binom{5}{2}}{\binom{10}{3}} = \frac{1}{6}$.

# Student

## STUDENT 2020 Q3

Rene marked two points a and b as accurately as possible on the number line. Which of the points p, q, r, s, t on the number line best represents their product ab?




(A) $p$ (B) $q$ (C) $r$ (D) $s$ (E) $t$

**Answer.** B

Since $a, b < 1$, so $ab < a < 1$, this excludes the points after a, which are $r$, $s$ and $t$. Moreover, $ab > \frac{1}{4}$ as $a, b > \frac{1}{2}$, so point $p$ is certainly to be excluded. This leaves the choice for point $q$.

## STUDENT 2020 Q20

A large integer N is divisible by all except two of the integers from 2 to 11. Which of the following pairs of integers could be these exceptions?

(A) 2 & 3 (B) 4 & 5 (C) 6 & 7 (D) 7 & 8 (E) 10 & 11

**Answer.** D

Since $\gcd(a, b) \nmid N$ for any positive integers $a \nmid N$ and $b \nmid N$ and vice versa, so we check separately for each choice given.

- If $2 \nmid N$ and $3 \nmid N$, then $6 \nmid N$, but, this contradicts the statement.
- If $4 \nmid N$ and $5 \nmid N$, then $8 \nmid N$, which contradicts the statement.
- If $6 \nmid N$ and $7 \nmid N$, then $2 \nmid N$ since $2 \mid 6$, but this contradicts the statement.
- If $10 \nmid N$ and $11 \nmid N$, then $2 \nmid N$ or $5 \nmid N$ as $2 \mid 10$ and $5 \mid 10$, but this contradicts the statement.

This leaves the possible integers be 7 and 8.

# Student

## STUDENT 2021 Q12

If $A = (0, 1) \cup (2, 3)$ and $B = (1, 2) \cup (3, 4)$, what is the set of all numbers of the form $a + b$ with $a$ in $A$ and $b$ in $B$?

(A) $(1, 7)$ (B) $(1, 5) \cup (5, 7)$ (C) $(1, 3) \cup (3, 7)$ (D) $(1, 3) \cup (3, 5) \cup (5, 7)$ (E) None of the previous

**Answer.** D

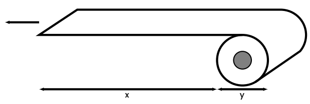
We consider the ranges of $a$ and $b$ separately.

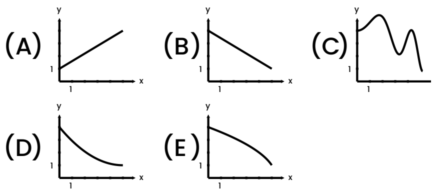
- Case 1: If $a \in (0, 1)$ and $b \in (1, 2)$, then $a + b \in (1, 3)$.
- Case 2: If $a \in (0, 1)$ and $b \in (3, 4)$ or $a \in (2, 3)$ and $b \in (1, 2)$, then $a + b \in (3, 5)$.
- Case 3: If $a \in (2, 3)$ and $b \in (3, 4)$, then $a + b \in (5, 7)$.

This yields that $a + b \in (1, 3) \cup (3, 5) \cup (5, 7)$ for $a \in A$ and $b \in B$.

## STUDENT 2020 Q19

A naughty pup grabs the end of a roll of toilet paper and walks away at a constant speed. Which of the functions below best describes the thickness y of the roll as a function of the unrolled part $x$?




**Answer.** D

Denote the inner radius, thickness and total length of the roll to be r, h and L respectively, while assume that the layers are strictly circular. The difference between a circle length and a corresponding spiral loop is negligible. Then the area between circles with radii $\frac{y}{2}$ and r is a side area of the paper tape, which is its length times thickness:

$$\pi \cdot \left[\left(\frac{y}{2}\right)^2 - r^2\right] = (L - x) \cdot h$$

Hence, $y = 2\sqrt{\frac{L - x}{\pi} h + r^2}$. The graph of y against x is sort of similar to $y = \alpha\sqrt{x_0 - \beta x}$ for some positive real some positive real numbers $x_0$, $\alpha$ and $\beta$.